Higher Maths: Unit 1: Algebraic functions and graphs

Student support notes

Transforming graphs

1. The basic transformations

Given the graph of $y = f(x)$, there are six basic transformations we can apply.
It is vital that you know each transformation and how it affects the original graph.
In particular, given any point **P** on the graph of $y = f(x)$, you should be able to write down the coordinates of its ***image***, **P′**, after a given transformation(s).

The table below summarises the six basic transformations as well as the effect of each on a given point **P** of the original graph.

New function	**Effect on** $y = f(x)$	**Effect on P**
$y = -f(x)$	The graph of $y = f(x)$ is reflected in the x-axis	**P** (x , y) → **P′**$(x , -y)$ y coordinates change sign
$y = f(-x)$	The graph of $y = f(x)$ is reflected in the y-axis	**P** (x , y) → **P′**$(-x , y)$ x coordinates change sign
$y = af(x)$	The graph of $y = f(x)$ is vertically stretched/compressed If **a > 1** then vertical stretch If **a < 1** then vertical compression	**P** (x , y) → **P′**(x , ay) y coordinates multiplied by a
$y = f(x) + d$	The graph of $y = f(x)$ is vertically translated If **d > 0** then translate up If **d < 0** then translate down	**P** (x , y) → **P′**$(x , y + d)$ Add d to all y coordinates
$y = f(x + c)$	The graph of $y = f(x)$ is horizontally translated If **c > 0** then translate left If **c < 0** then translate right	**P** (x , y) → **P′**$(x - c , y)$ x coordinates change
$y = f(bx)$	The graph of $y = f(x)$ is horizontally stretched/compressed If **b > 1** then horizontal compression If **0 < b < 1** then horizontal stretch	**P** (x , y) → **P′**$(x/b, y)$ x coordinates scaled

Often you will be given the graph of $y = f(x)$ and you will have to sketch the graph of a transformed function.

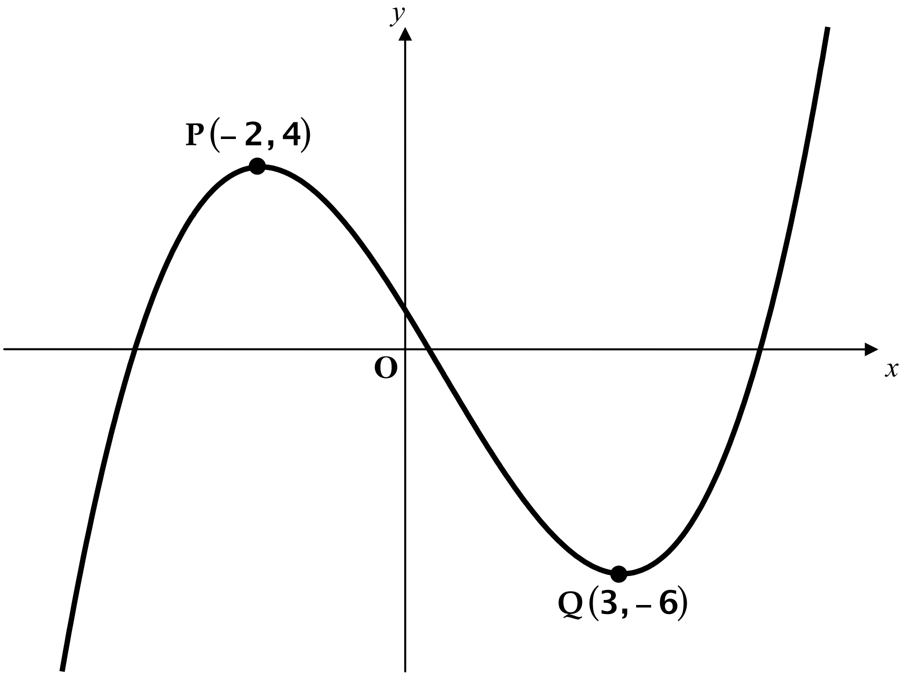
2. Basic examples

Examples 1 to 6 all refer to the same cubic function $y = f(x)$, as shown below.

Example 1

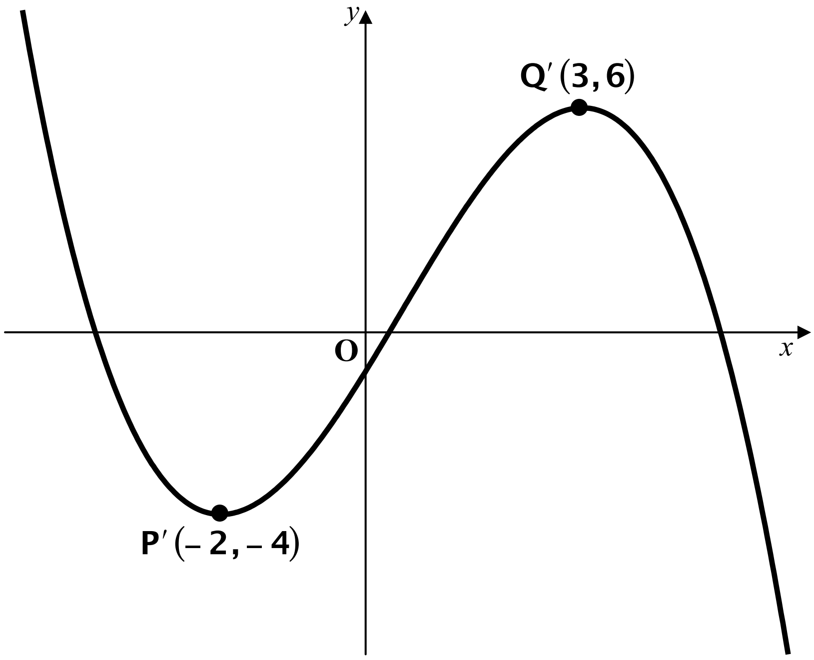
The diagram below shows part of the graph of a cubic function $y = f(x)$. The graph has turning points at P $(-2, 4)$ and Q $(3, -6)$, respectively.

Sketch the graph of $y = -f(x)$.



Solution

To make the graph of $y = -f(x)$ we have to reflect the graph of $y = f(x)$ in the x-axis.



Make sure that when you are sketching a transformed graph you get the shape of the graph right and also label it correctly.

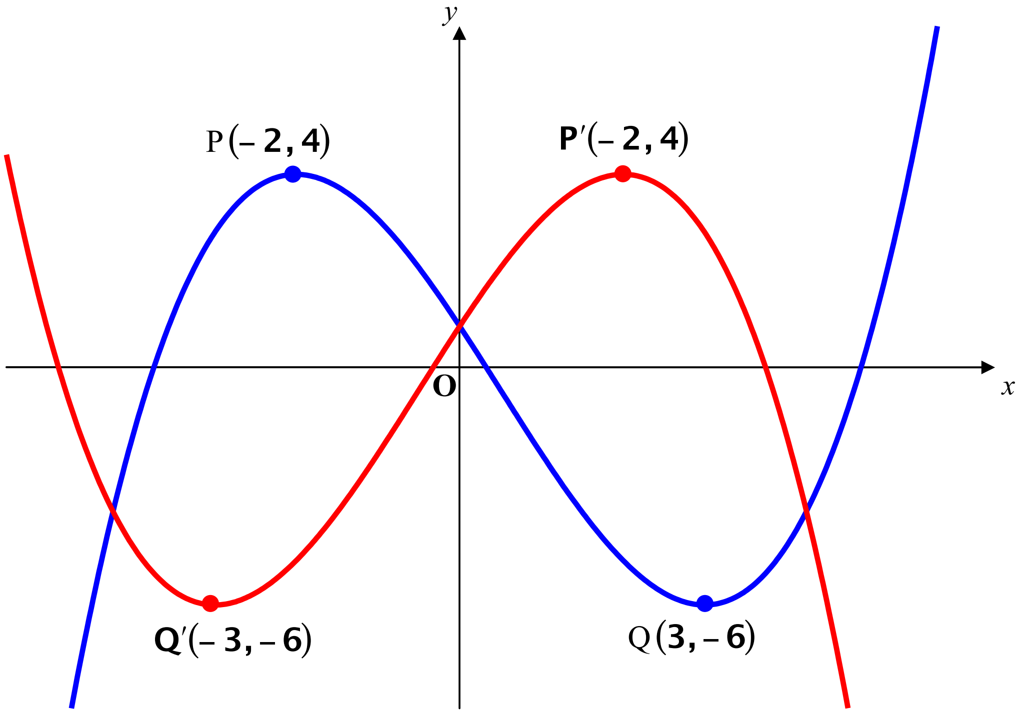
Example 2

Sketch the graph of $y = f(-x)$.

Solution

To make the graph of $y = f(-x)$ we have to reflect the graph of $y = f(x)$ in the y-axis.

The graphs of $y = f(x)$ and $y = f(-x)$ are shown below.



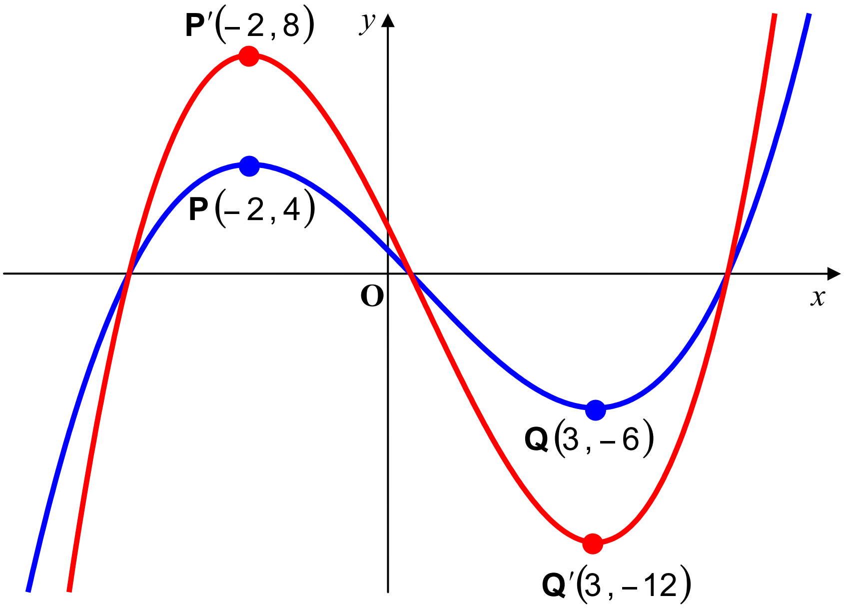
Example 3

Sketch the graph of $y = 2f(x)$.

Solution

To make the graph of $y = 2f(x)$ we have to vertically stretch the graph of $y = f(x)$.

The graphs of $y = f(x)$ and $y = 2f(x)$ are shown below.



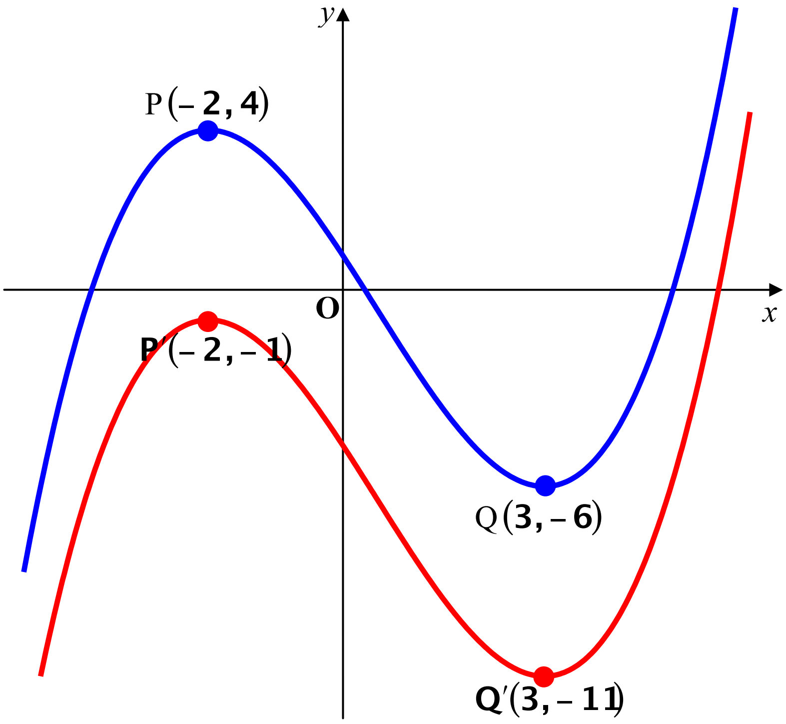
Example 4

Sketch the graph of $y = f(x) - 5$.

Solution

To make the graph of $y = f(x) - 5$ we have to vertically translate the graph of $y = f(x)$. Since $d = -5$ we have to move the graph of $y = f(x)$ vertically downwards.

The graphs of $y = f(x)$ and $y = f(x) - 5$ are shown below.



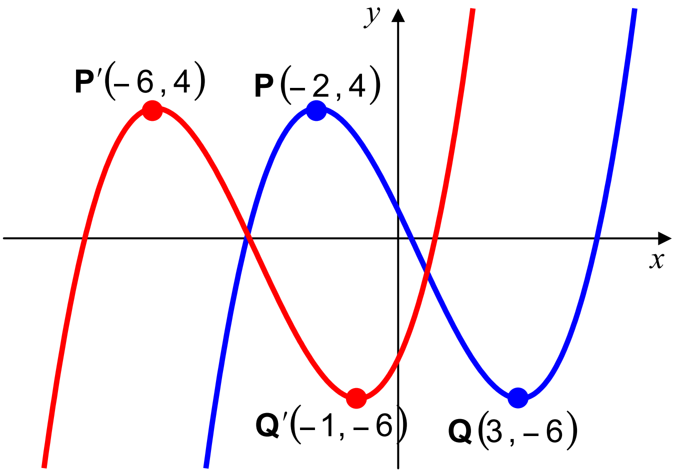
Example 5

Sketch the graph of $y = f(x+4)$.

Solution

To make the graph of $y = f(x+4)$ we have to horizontally translate the graph of $y = f(x)$. Since c is positive we have to move the graph of $y = f(x)$ to the left.

The graphs of $y = f(x)$ and $y = f(x+4)$ are shown below.

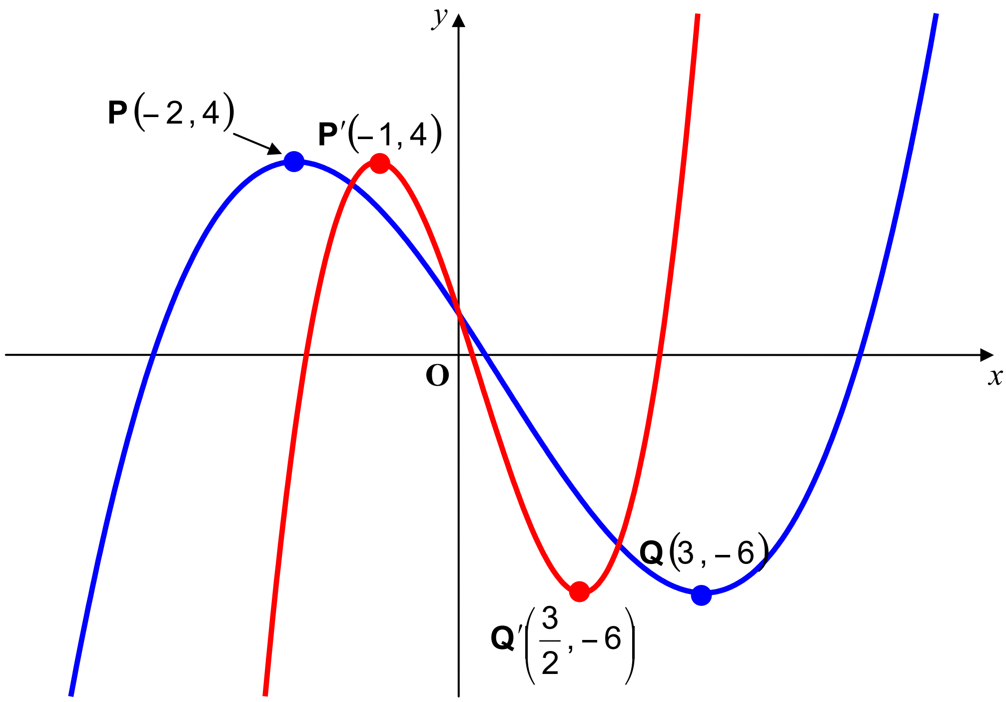


Example 6

Sketch the graph of $y = f(2x)$.

Solution

To make the graph of $y = f(2x)$ we have to horizontally compress the graph of $y = f(x)$. The graphs of $y = f(x)$ and $y = f(x+4)$ are shown below.



2. Harder examples (more than one transformation)

Sometimes you may need to carry out more than one transformation of the graph of $y = f(x)$. Again, it is important that you correctly identify each transformation and, if necessary, carry out the transformations in the correct order.

There are three situations to be aware of:

1. **Vertical transformations only** eg $y = f(x) \rightarrow y = af(x) + d$

   You should deal with *a* followed by *d* (stretch then translate) (think about the order of operations)

2. **Horizontal transformations only** eg $y = f(x) \rightarrow y = f(bx + c)$

   You should deal with *c* followed by *b* (translate then compress/stretch)

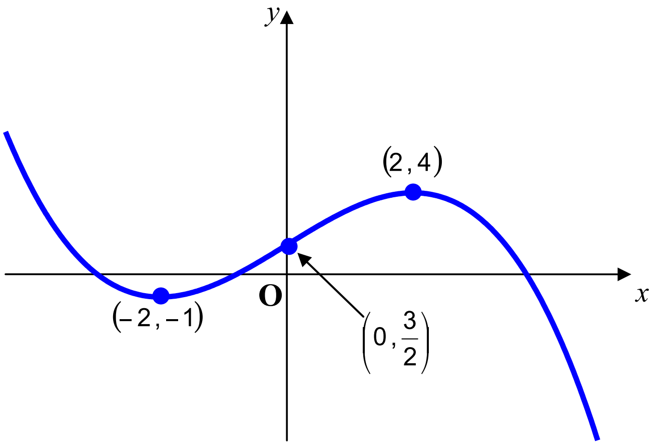
3. **A mixture** eg $y = f(x) \rightarrow y = af(bx)$

   You can deal with the horizontal transformations and then the vertical ones or vice versa.

   Remember that in this situation if you have more than one vertical or horizontal transformation you should prioritise using 1 and 2.

Example 1

The diagram below shows part of the graph of a cubic function $y = f(x)$. The graph passes through the points $(-2, -1)$, $\left(0, \frac{3}{2}\right)$ and $(2, 4)$. Sketch the graph of $y = 3f(x) - 2$.



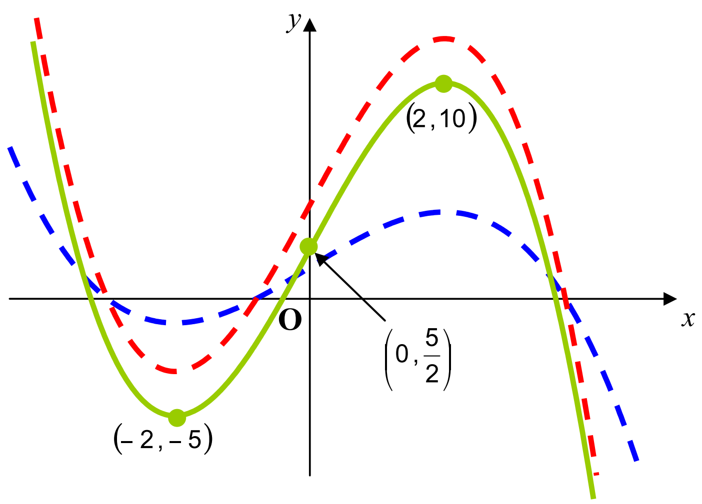
Solution

Comparing the new function with $y = af(bx + c) + d$ we see that $a = 3$ and $d = -2$. This tells us that we are only performing vertical transformations on the original graph.

Firstly, we vertically stretch the graph of $y = f(x)$ by a factor of 3 and then vertically translate the resultant graph by -2 units.

After the first transformation $y = f(x) \rightarrow y = 3f(x)$. The images of the three given points on $y = f(x)$ are $(-2, -3)$, $\left(0, \frac{9}{2}\right)$ and $(2, 12)$.

After the second transformation $y = 3f(x) \rightarrow y = 3f(x) - 2$. The images of the three given points on $y = f(x)$ are now $(-2, -5)$, $\left(0, \frac{5}{2}\right)$ and $(2, 10)$.

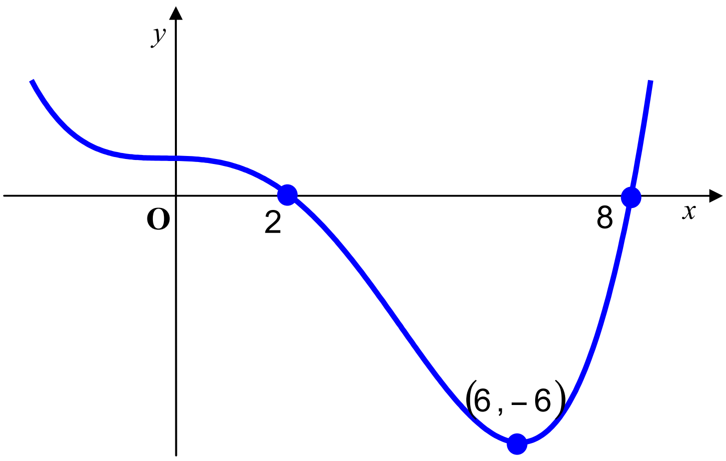


Example 2

The diagram below shows part of the graph of the function $y = f(x)$. The graph crosses the x-axis at the points $(2, 0)$ and $(8, 0)$. The graph has a minimum turning point at $(6, -6)$.

The function g is defined by $g(x) = f(2x - 4)$.

Sketch the graph of $y = g(x)$.



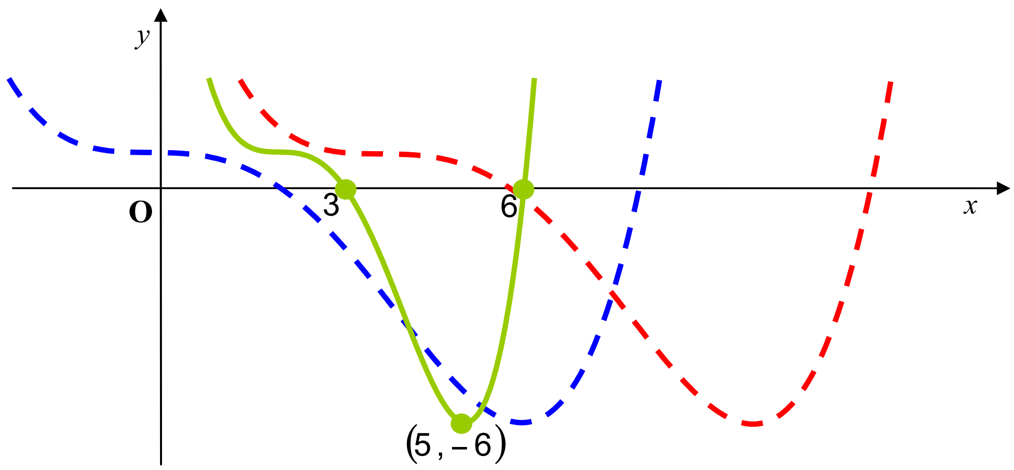
Solution

Comparing the new function with $y = af(bx + c) + d$ we see that $b = 2$ and $c = -4$. This tells us that we are only performing horizontal transformations on the original graph.

Firstly, we horizontally translate the graph of $y = f(x)$ by +4 units and then horizontally compress the resultant graph by a factor of 2.

After the first transformation $y = f(x) \rightarrow y = f(x - 4)$. The images of the three given points on $y = f(x)$ are $(6, 0)$, $(10, -6)$ and $(12, 0)$.

After the second transformation $y = f(x - 4) \rightarrow y = f(2x - 4)$. The images of the three given points on $y = f(x)$ are now $(3, 0)$, $(5, -6)$ and $(6, 0)$.

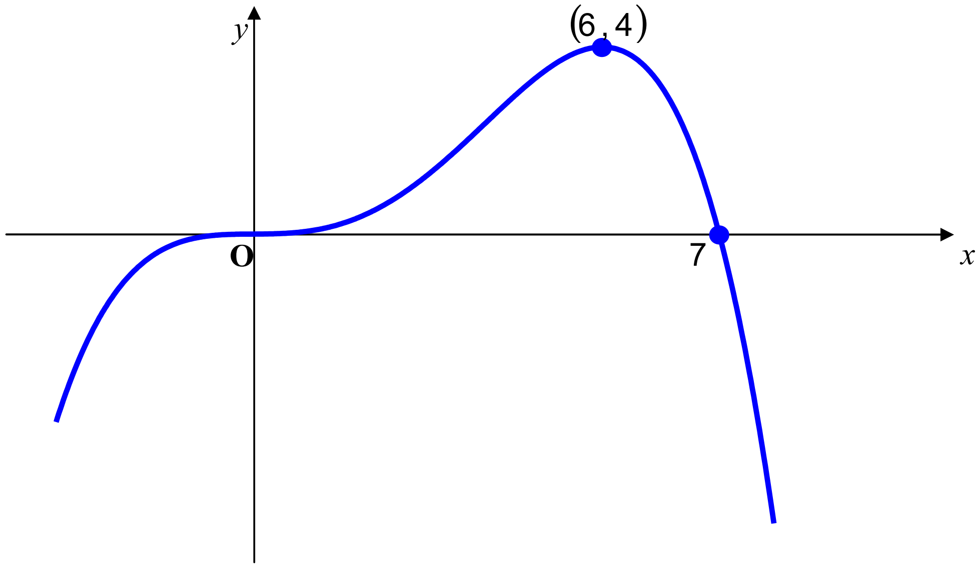


Be especially careful with horizontal translations. When c is positive the graph moves to the left. When c is negative the graph moves to the right.

Example 3

The diagram below shows part of the graph of the function $y = f(x)$. This graph passes through the origin, crosses the x-axis at $(7, 0)$ and has a maximum turning point at $(6, 4)$.

Sketch the graph of $y = -2f(x+3)$.



Solution

Comparing the new function with $y = af(bx + c) + d$ we see that $a = -2$ and $c = 3$. This tells us that there is a mixture of horizontal and vertical transformations. We can carry these out in any order.

Starting with the vertical transformations:

1. $y = f(x) \rightarrow y = -f(x)$ (reflect the graph of $y = f(x)$ in the *x*-axis.)
2. $y = -f(x) \rightarrow y = -2f(x)$ (vertically stretch the resultant graph by a factor of 2.)

Note that you could have vertically stretched and then reflected in the *x*-axis.

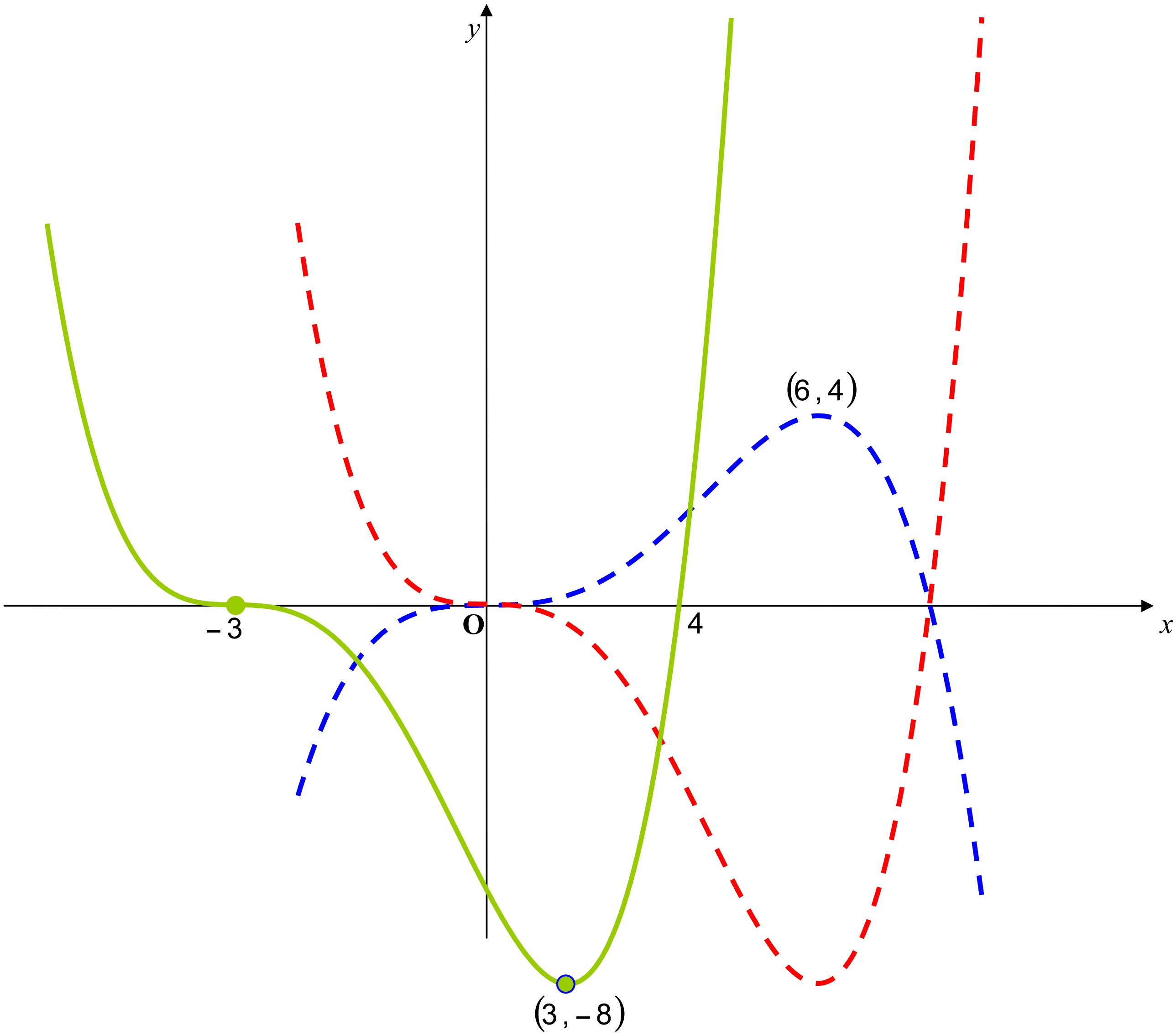
The images of the three given points on $y = f(x)$ are $(0, 0)$, $(6, -8)$ and $(7, 0)$.

Now we can carry out the horizontal transformation:

$y = -2f(x) \rightarrow y = -2f(x + 3)$ (horizontally translate by –3 units, ie to the left)

The images of the three given points on $y = f(x)$ are now $(0, 0)$, $(3, -8)$ and $(4, 0)$.

The sketches are shown below.



y
x
O
−3
4
(6 , 4)
(3 , −8)

Example 4

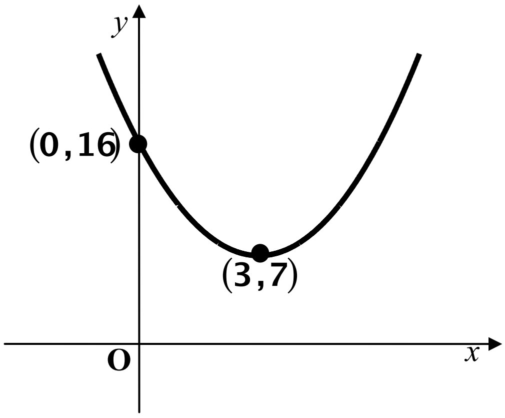
The diagram opposite shows the graph of a quadratic function $y = f(x)$. It has a minimum turning point at $(3,7)$ and meets the y-axis at the point $(0,16)$.

The function h is defined by $h(x) = 16 - f(x)$.

(a) On the same diagram, sketch the graph of $y = h(x)$.

(b) Find the values of x for which $16 - f(x) \leq 0$.

(c) Express $h(x)$ in terms of x.



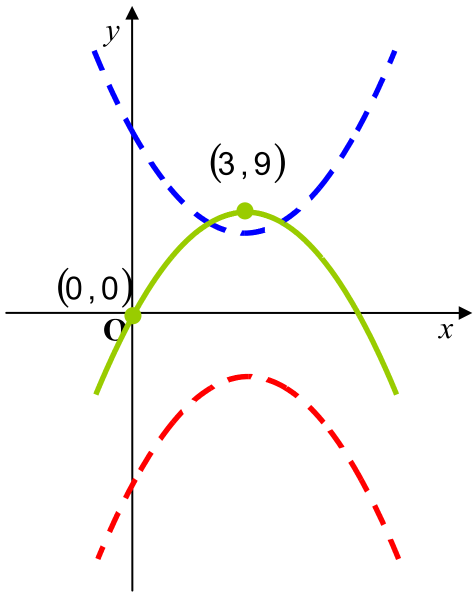
Solution (a)

Firstly note that $h(x) = -f(x) + 16$. Changing the order of terms on the right-hand side makes it easier to see how to transform the graph of $y = f(x)$. Comparing the new function with $y = af(bx + c) + d$ we see that $a = -1$ and $d = 16$. This tells us that we are only performing vertical transformations on the original graph.

Firstly, we reflect the graph of $y = f(x)$ in the x-axis and then vertically translate the resultant graph by $+16$ units.

After the first transformation $y = f(x) \to y = -f(x)$. The images of the two given points on $y = f(x)$ are $(0, -16)$ and $(3, -7)$.

After the second transformation $y = -f(x) \to y = -f(x) + 16$. The images of the three given points on $y = f(x)$ are now $(0, 0)$ and $(3, 9)$.



Solution (b)

Note by symmetry that the graph of $y = h(x)$ will also meet the x-axis at $(6, 0)$.

$16 - f(x) \le 0 \Rightarrow h(x) \le 0$, ie we are searching for points on the graph that are on or below the x-axis.

To see why, remember that any point on the graph of $y = h(x)$ has coordinates $(x, h(x))$. $h(x) \le 0$ means that the y-coordinates of any point must be less than or equal to zero. Looking at the graph we can easily see that for $h(x) \le 0$, $x \le 0$ or $x \ge 6$.

Solution (c)

Don't be confused by the wording. You are simply being asked to find the equation for $h(x)$.

The function $y = h(x)$ is quadratic so you need to remember the three different ways in which the equation can be worked out (refer to your notes on quadratics). Since we know the roots of the graph and also the coordinates of the turning point, we are free to use any of the three methods.

Here we will use the root form of the equation. The roots are 0 and 6 and so we have:

$$\begin{aligned} h(x) &= a(x-0)(x-6) \\ \Rightarrow \quad h(x) &= ax(x-6) \end{aligned}$$

To find the value of a, use the fact that $(3, 9)$ is on the graph. This tells us that when $x = 3$, $h(x) = 9$. (Remember $(x, h(x))$).

Substituting into the equation gives:

$$\begin{aligned} h(x) &= ax(x-6) \\ 9 &= a \times 3 \times (3-6) \\ 9 &= -9a \\ \Rightarrow \quad a &= -1 \end{aligned}$$

We now have:

$$h(x) = -1x(x-6)$$

Expanding the brackets gives:

$$h(x) = -x^2 + 6x = 6x - x^2$$